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On the invalidity of Dirac's conjecture for a system with a singular higher-order Lagrangian

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Abstract
Based on the canonical Nöther's theorem and Poincaré–Cartan integral invariant for a system with a singular higher-order Lagrangian, we present a counterexample with any higher-order derivatives, in which no linearizations of constraints are made to the system, showing that Dirac's conjecture is invalid.

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1. Introduction

Local gauge invariance is now a central concept in modern field theories. There are some constraints in phase space in the gauge theories that follow the requirement of local gauge invariance. At present, Dirac's theory of constrained systems [1] plays an important role in theoretical physics. Dirac's canonical formalism is the foundation of most general quantization methods of gauge theories and has succeeded greatly in recent years. Nowadays the so-called BRST (Beechi, Rouet, Stora, Tyutin) or BFV (Batalin, Fradkin, Vilkoviski) procedures are of special importance. They allow us to perform the quantization of general gauge and gravitational theories in a consistent way, not only with the Feynman path integral approach but also at the operator level. Many quantization procedures of the gauge and gravitational fields have been found.

However, in spite of its great achievements in many fields, some basic problems about the theory itself are still widely discussed in the literature, one of which being Dirac's conjecture [1]. In his work on the generalized canonical formalism, Dirac conjectured that all first-class constraints are independent generators of gauge transformations that generate equivalent transformations among physical states. For both the Faddeev–Senjanovic procedure [2] and BFV procedure based on BRST symmetry [3], the gauge conditions are always directly related to the first-class constraints. If Dirac's conjecture were invalid, the number of gauge conditions would change [4]. Thus the validity of Dirac's conjecture is of great importance to the quantization of the system [4].

The question whether Dirac's conjecture is valid is closely connected with the problem of the possible equivalence between Dirac's procedure in terms of the extended Hamiltonian H_E and the Lagrangian description [5, 6]. There have been objections to Dirac's conjecture from time to time [7–10]. Several examples [11–13] were provided, where constraints are re-written in linearized form, indicating that Dirac's conjecture is invalid. But it has been argued [14] that in those models the apparent failure of Dirac's conjecture resulted from the improper linearization of the function of the secondary first-class constraints (SFCC). If we avoid rewriting those constraints in a linearized form and re-examine those supposed counterexamples, we can show that Dirac's conjecture is still valid [14]. On the other hand, we have also provided some counterexamples for a system with a singular first-order Lagrangian [15, 16], in which Dirac's conjecture fails. Our counterexamples are discussed without the linearization to the constraints. Here we discuss the invalidity of Dirac's conjecture for a system with a singular higher-order Lagrangian.

2. Dirac's conjecture for systems with singular higher-order Lagrangian

Let us consider a system whose Lagrangian is given by

$$L = L(t; q_{(0)}, q_{(1)}, q_{(2)}, \ldots, q_{(N)})$$

where

$$q = [q^1, q^2, \ldots, q^N] \qquad q_{(0)} = q \qquad q_{(s)} = \frac{\mathrm{d}^s}{\mathrm{d}t^s} q(t).$$

According to the Ostrogradsky transformation, we introduce the generalized canonical momentum

$$p_i^{(N-1)} = \frac{\partial L}{\partial q^i_{(N)}} \qquad (i = 1, 2, \ldots, N) \tag{1a}$$

$$p_i^{(s-1)} = \frac{\partial L}{\partial q^i_{(s)}} - \dot{p}_i^{(s)} \qquad (s = 1, 2, \ldots, N-1). \tag{1b}$$

The canonical Hamiltonian is

$$H_c = \sum_{s=0}^{N-1} p_i^{(s)} q^i_{(s+1)} - L \tag{2}$$

which can be formed by eliminating the highest-order derivatives $q^i_{(N_i)}$ from (1a). When the extended Hessian matrix $[\partial^2 L/\partial q^j_{(N)}]$ is degenerate, we cannot solve all $q^i_{(N)}$ from (1a). This implies that there are constraints in phase space. Suppose the primary first-class constraints (PFCC) of the system are $\Lambda_a \approx 0$ $(a = 1, 2, \ldots, K)$, and the second-class constraints of the system are $\theta_i \approx 0$ $(i = 1, 2, \ldots, I)$. For any mechanical quantity $F(t; q, p)$, we have [17]

$$\frac{\mathrm{d}F}{\mathrm{d}t} \approx \frac{\partial F}{\partial t} + \{F, H_c\} + \lambda^\alpha \{F, \Lambda_\alpha\} + \{F, \theta_{b'}\} C^{-1}_{b'b} \left[\{\theta_b, H_c\} + \frac{\partial \theta_b}{\partial t}\right] \tag{3}$$

where constraint multipliers λ^α related to first-class constraints are arbitrary functions, and $C^{-1}_{b'b}$ satisfy $C^{-1}_{b'b}\{\theta_b, \theta_{b''}\} = \delta_{b'b''}$. Let δt be a infinitesimal parameter, then

$$F(\delta t) = F(0) + \delta t \left(\frac{\partial F}{\partial t} + \{F, H_c\} + \lambda^a \{F, \Lambda_a\} + \{F, \theta_{b'}\} C^{-1}_{b'b} \left[\{\theta_b, H_c\} + \frac{\partial \theta_b}{\partial t}\right]\right)_{t=0}. \tag{4}$$

Choosing another multiplier $\tilde{\lambda}^a(t)$, we can get an expression similar to (4), with λ^a replaced by $\tilde{\lambda}^a$. Subtracting these two results relevant to λ^a and $\tilde{\lambda}^a$, respectively, we get

$$\delta F = \varepsilon^a \{F, \Lambda_a\} \tag{5}$$

where

$$\varepsilon^a = \delta t[\lambda^a(0) - \tilde{\lambda}^a(0)]. \tag{6}$$

Equation (5) shows that PFCC are generators of gauge transformations. Let $\omega^{a'}$ be another transformation parameter like ε^a. Now we do the transformation twice according to equation (4), firstly with ε^a, and then with $\omega^{a'}$. We get

$$\tilde{F} = F(0) + \varepsilon^a\{F, \Lambda_a\} + \omega^{a'}\{F(0) + \varepsilon^a\{F, \Lambda_a\}, \Lambda_{a'}\}.$$

Then we reverse the sequence of transformations, firstly with $\omega^{a'}$ and then with ε^a:

$$\tilde{F} = F(0) + \omega^{a'}\{F, \Lambda_{a'}\} + \varepsilon^a\{F(0) + \omega^{a'}\{F, \Lambda_{a'}\}, \Lambda_a\}.$$

Subtracting these two results, we get

$$\delta F = \varepsilon^a \omega^{a'} \{F, \{\Lambda_a, \Lambda_{a'}\}\} \tag{7}$$

where Jacobi's identity of a generalized Poisson bracket is used. Now we can see that the Poisson bracket of two primary first-class constraints is also a generator of gauge transformation. The Poisson bracket of two primary first-class constraints is still a first-class constraint. $\{\Lambda_a, \Lambda_{a'}\}$ could be either a PFCC or SFCC. Thus one can extend Dirac's conjecture to the higher-order singular Langrangian system containing time explicitly: all first-class constraints are generators of gauge transformations, generating equivalent transformations among physical states.

3. The invalidity of Dirac's conjecture for a system with a singular higher-order Lagrangian

If Dirac's conjecture holds true, the dynamics of a system possessing PFCC and SFCC should be correctly described by the equations of motion deriving from the extended Hamiltonian [5,6]

$$H_E = H_c + \lambda^a \phi_a^0 + \mu_k^a \phi_a^k \tag{8}$$

where H_c is a canonical Hamiltonian, ϕ_a^0 are PFCC, ϕ_a^k are SFCC, and λ^a and μ_k^a are corresponding constraint multipliers, respectively. We can study the conservation laws derived from H_E via the canonical Nöther first theorem to discuss the validity of Dirac's conjecture [18]. Let us consider a model whose Lagrangian is given by

$$L = \sum_{s=1}^{N} (x_{(s)} z_{(s)} - y_{(s-1)} z_{(s)}) + xz. \tag{9}$$

The Lagrangian (9) is invariant under scale transformation

$$\begin{aligned} &x'_{(s)} = \rho^{-1} x_{(s)} \qquad y'_{(s)} = \rho^{-1} y_{(s)} \qquad z'_{(s)} = \rho z_{(s)} \\ &p_x^{\prime(N-1)} = \rho p_x^{(N-1)} \qquad p_z^{\prime(N-1)} = \rho^{-1} p_z^{(N-1)} \end{aligned} \tag{10}$$

where ρ is a numerical parameter. According to the generalized canonical first Nöther's theorem, one can obtain the conservation law

$$p_x^{(s)} x_{(s)} + p_y^{(s)} y_{(s)} - p_z^{(s)} z_{(s)} = \text{const.} \tag{11}$$

The situation when $N = 1, 2$ has been discussed [10, 18, 19], where Dirac's conjecture is pointed out to be invalid. Following the same method, let $N = 3$ in (9). The canonical momentum are given by

$$p_x^{(2)} = z_{(3)} \qquad p_y^{(2)} = 0 \qquad p_z^{(2)} = x_{(3)} - y_{(2)} \tag{12}$$

and the canonical Hamiltonian is given by

$$\begin{aligned} H_c &= \sum_{s=0}^{2} p_i^{(s)} q_{(s+1)}^i - L \\ &= p_x^{(2)}(p_z^{(2)} + y_{(2)}) + \sum_{s=0}^{1} p_i^{(s)} q_{(s+1)}^i - \tilde{L} \end{aligned} \tag{13}$$

where $\tilde{L} = \sum_{s=1}^{2}(x_{(s)}z_{(s)} - y_{(s-1)}z_{(s)}) + xz$. The primary constraint is

$$\phi^0 = p_y^{(2)} \approx 0. \tag{14}$$

The stationary condition of the constraint yields the following secondary constraints:

$$\begin{aligned} \phi^1 &= \{\phi^0, H_T\} = -p_x^{(2)} - p_y^{(1)} \approx 0 \\ \phi^2 &= \{\phi^1, H_T\} = p_x^{(1)} + p_y^{(0)} \approx 0 \\ \phi^3 &= \{\phi^2, H_T\} = -p_x^{(0)} \approx 0 \\ \phi^4 &= \{\phi^3, H_T\} = -z \approx 0 \\ \phi^5 &= \{\phi^4, H_T\} = -z_{(1)} \approx 0 \\ \phi^6 &= \{\phi^5, H_T\} = -z_{(2)} \approx 0 \\ \phi^7 &= \{\phi^6, H_T\} = -p_x^{(2)} \approx 0 \\ \phi^8 &= \{\phi^7, H_T\} = p_x^{(1)} - z_{(2)} \approx 0. \end{aligned}$$

All the constraints are first class. If Dirac's conjecture holds true, the dynamics of this system should be described by the equations of motion arising from the extended Hamiltonian (8). All SFCC in the Hamiltonian are taken into account. According to the generalized canonical first Nöther's theorem, the existence of the conservation law (11) require that all SFCC $\phi^k \approx 0\,(k = 1, 2, \ldots, 8)$ be invariant under transformation (10). But it is clear that the above constraints cannot satisfy these conditions under transformation (10). Thus the conservation law (11) could not be obtained from the extended Hamiltonian (8). This implies that Dirac's conjecture is invalid in this model.

If we let $N = 4, 5, 6$ in (9) and repeat these procedures, we can get similar results: Dirac's conjecture fails in all these examples. The total number of secondary constraints derived is 13 (when $N = 4$), 16 (when $N = 5$) and 19 (when $N = 6$). For the convenience of further discussion, we give the result in detail for $N = 4$ in (9). All the constraints are first class, and they are given by

$$\begin{aligned} \phi^0 &= p_y^{(2)} \approx 0 \\ \phi^1 &= \{\phi^0, H_T\} = -p_x^{(3)} - p_y^{(2)} \approx 0 \\ \phi^2 &= \{\phi^1, H_T\} = p_x^{(2)} + p_y^{(1)} \approx 0 \\ \phi^3 &= \{\phi^2, H_T\} = -p_x^{(1)} - p_y^{(0)} \approx 0 \\ \phi^4 &= \{\phi^3, H_T\} = p_x^{(0)} \approx 0 \\ \phi^5 &= \{\phi^4, H_T\} = z_{(0)} \approx 0 \\ \phi^6 &= \{\phi^5, H_T\} = z_{(1)} \approx 0 \end{aligned}$$

$$\begin{aligned}
\phi^7 &= \{\phi^6, H_T\} = z_{(2)} \approx 0 \\
\phi^8 &= \{\phi^7, H_T\} = z_{(3)} \approx 0 \\
\phi^9 &= \{\phi^3, H_T\} = p_x^{(3)} \approx 0 \\
\phi^{10} &= \{\phi^4, H_T\} = -p_x^{(2)} + z_{(3)} \approx 0 \\
\phi^{11} &= \{\phi^5, H_T\} = p_x^{(1)} - z_{(2)} + p_x^{(3)} \approx 0 \\
\phi^{12} &= \{\phi^6, H_T\} = -p_x^{(0)} + z_{(1)} - p_x^{(2)} \approx 0 \\
\phi^{13} &= \{\phi^7, H_T\} = p_x^{(1)} - z_{(0)} \approx 0.
\end{aligned}$$

Now we study the general situation (9)

$$L = \sum_{s=1}^{N} (x_{(s)} z_{(s)} - y_{(s-1)} z_{(s)}) + xz.$$

Following the same procedures, we have the canonical momentum

$$\begin{aligned}
&p_x^{(N-1)} = z_{(N)} \qquad p_y^{(N-1)} = 0 \\
&p_z^{(N-1)} = x_{(N)} - y_{(N-1)}.
\end{aligned} \tag{15}$$

The canonical Hamiltonian is

$$H_c = \sum_{s=0}^{N-1} p_i^{(s)} q_{(s+1)}^i - L = p_x^{(N-1)} (p_z^{(N-1)} + y_{(N-1)}) + \sum_{s=0}^{N-2} p_i^{(s)} q_{(s+1)}^i - \tilde{L} \tag{16}$$

where $\tilde{L} = \sum_{s=1}^{N-1} (x_{(s)} z_{(s)} - y_{(s-1)} z_{(s)}) + xz$. For any positive integer N, the primary constraint is

$$\phi^0 = p_y^{(N-1)} \approx 0. \tag{17}$$

For integer $N > 3$, we have

$$\begin{aligned}
\phi^1 &= \{\phi^0, H_T\} = -p_x^{(N-1)} - p_y^{(N-2)} \approx 0 \\
\phi^2 &= \{\phi^1, H_T\} = p_x^{(N-2)} + p_y^{(N-3)} \approx 0 \\
\phi^3 &= \{\phi^2, H_T\} = -p_x^{(N-3)} - p_y^{(N-4)} \approx 0
\end{aligned} \tag{18}$$

. . . .

$$\phi^m = \{\phi^{m-1}, H_T\} = (-1)^m p_x^{(N-m)} + (-1)^m p_y^{(N-m-1)} \approx 0 \tag{19}$$

. . . .

$$\begin{aligned}
\phi^{N-1} &= \{\phi^{N-2}, H_T\} = (-1)^{N-1} p_x^{(1)} + (-1)^{N-1} p_y^{(0)} \approx 0 \\
\phi^N &= \{\phi^{N-1}, H_T\} = (-1)^N p_x^{(0)} \approx 0 \\
\phi^{N+1} &= \{\phi^N, H_T\} = (-1)^N z_{(0)} \approx 0 \\
\phi^{N+2} &= \{\phi^{N+1}, H_T\} = (-1)^N z_{(1)} \approx 0 \\
\phi^{N+3} &= \{\phi^{N+2}, H_T\} = (-1)^N z_{(2)} \approx 0
\end{aligned} \tag{20}$$

. . . .

$$\begin{aligned}
\phi^{2N} &= \{\phi^{2N-1}, H_T\} = (-1)^N z_{(N-1)} \approx 0 \\
\phi^{2N+1} &= \{\phi^{2N}, H_T\} = (-1)^N p_x^{(N-1)} \approx 0 \\
\phi^{2N+2} &= \{\phi^{2N+1}, H_T\} = (-1)^{N+1} p_x^{(N-2)} + (-1)^N z_{(N-1)} \approx 0 \\
\phi^{2N+3} &= \{\phi^{2N+2}, H_T\} = (-1)^{N+2} p_x^{(N-3)} + (-1)^{N+1} z_{(N-2)} + (-1)^N p_x^{(N-1)} \approx 0 \\
\phi^{2N+4} &= \{\phi^{2N+3}, H_T\} = (-1)^{N+3} p_x^{(N-4)} + (-1)^{N+2} z_{(N-3)} + (-1)^{N+1} p_x^{(N-2)} \approx 0
\end{aligned} \tag{21}$$

. . . .

$$\phi^{2N+k} = \{\phi^{2N+k-1}, H_T\} \\ = (-1)^{N+k-1} p_x^{(N-k)} + (-1)^{N+k-2} z_{(N-k+1)} + (-1)^{N+k-3} p_x^{(N-k+2)} \approx 0 \tag{22}$$

. . . .

$$\begin{aligned} \phi^{3N-1} &= \{\phi^{3N-2}, H_T\} = (-1)^{2N-2} p_x^{(1)} + (-1)^{2N-3} z_{(2)} + (-1)^{2N-4} p_x^{(3)} \\ &= p_x^{(1)} - z_{(2)} + p_x^{(3)} \approx 0 \\ \phi^{3N} &= \{\phi^{3N-1}, H_T\} = -p_x^{(0)} + z_{(1)} - p_x^{(2)} \approx 0 \\ \phi^{3N+1} &= \{\phi^{3N}, H_T\} = -z_{(0)} + p_x^{(1)} \approx 0. \end{aligned} \tag{23}$$

Then we get $\{\phi^{3N}, H_T\} = -p_x^{(0)} = (-1)^{(N-1)}\phi^N$, which is not another constraint. Therefore we have found all constraints for the system (9), and all of them are first-class constraints. We can see that some secondary constraints (for example, $z_{(0)} \approx 0$) are not invariant under (10). It means Dirac's conjecture fails for this system.

Thus we can see that Dirac's conjecture always fails for a system whose Lagrangian is given by (9), letting N be any positive integer. The total number of secondary constraints contained in the system is generally $3N + 1$ (13 for $N = 3$ (shown above) and 16 for $N = 4$, respectively). The exceptions are that, when $N < 3$, because N is not large enough to go through all the steps for the general case (18)–(23), the total number of secondary constraints is not $3N+1$. It has been calculated that the number of secondary constraints for those exceptions are 2 for $N = 1$ [19], 3 for $N = 2$ [10, 18] and 8 for $N = 3$ (see above). Therefore we can conclude that Dirac's conjecture is generally invalid for singular higher-order Lagrangian systems.

4. Poincaré–Cartan integral invariant and Dirac's conjecture

The above conclusion could also be obtained using generalized the Poincaré–Cartan integral invariant for a system with a singular higher-order Lagrangian. As is well known, the Poincaré–Cartan integral invariant is equivalent to the equation of motions in classical mechanics. the Poincaré–Cartan integral invariant has been generalized to a system with a singular higher-order Lagrangian [10,20]. If the dynamical of a system is generated by an extended Hamiltonian H_E obtained by adding a linear combination of all SFCC to the PFCC, the generalized Poincaré–Cartan integral invariant can also be deduced, as long as all SFCC are invariant under the total variation of canonical variables, including time. The use of the generalized Poincaré–Cartan integral invariant enables one to write the equation of motion arising from the extended Hamiltonian H_E, and we can see that all first-class constraints appear in the Hamiltonian, where one cannot introduce any distinction between PFCC and SFCC. That is to say, the existence of the Poincaré–Cartan integral invariant for a system with a singular higher-order Lagrangian implies that Dirac's conjecture holds true for the system. The necessary and sufficient condition for the equations to be the generalized canonical equations arising from the extended Hamiltonian H_E is that the generalized Poincaré–Cartan integral invariant exists for such a system. The generalized Poincaré–Cartan integral invariant for a system with a singular higher-order Lagrangian differs from the usual one having a regular Lagrangian in that the constraints must be invariant under the total variation of canonical variables, including time. For the model mentioned above, the constraints $\phi^k \approx 0$ ($k = 1, 2, \ldots, 3N+1$) should be invariant under the transformation (10). Owing to these restrictions on canonical variables, one cannot deduce all the generalized canonical equations arising from the extended Hamiltonian H_E with the generalized Poincaré–Cartan integral invariant. This implies that the equivalence between the generalized Poincaré–Cartan integral invariant and the generalized canonical equations derived from H_E is violated. Thus we can find that Dirac's conjecture is invalid for a system with a singular higher-order Lagrangian. It should be noticed that there is no linearization to the constraints in this model.

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